COMPETITION PROBLEMS & MATH WRANGLES:
Adventures in Actively Using Math

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Why competitions?

Competitions take thought, make use of broad problem-solving strategies, and requires the ability to think through problems that don't have the benefit of context provided by a textbook or classroom instruction.

From SFPS SAGE administrator and teacher Geoffrey Moon: "The roots 'con' and 'com' mean together. Contests and competitions bring students together to enjoy a common challenge, encouraging each other, and sharing perspectives on the way they each have approached something difficult. Young or novice contestants who score no points in a contest look to those who do well and develop a sense of who they can become. Experienced and successful contestants experience pride in being able accomplish something special. In math competition, talented mathletes develop a sense of community that cannot be found anywhere else."

Metrics, competitions, and visibility: One of the appealing aspects of competitions is that, for better or worse, it is a measure of something. What competitions can do, then, is make visible your efforts to improve the math culture at your school and in your district via a clear external metric. And it is true that some kids just love competing and challenging themselves against cool problems!

Why math wrangles?

In a limited way, math wrangles are a kind of competition. In other ways, they are richer, deeper ways to engage students in both mathematical thought and communication. And unlike competitions, nearly all students find math wrangles *fun*!

A math wrangle is a *mathematics debate*. This is not as strange as it sounds. In its standard form, teams of students go off into rooms to study a set of eight problems, then return to the debate setting and the coin toss. The winning toss permits the team to challenge the opponents to solve one of the problems. That team may accept, and a member presents their solution, deals with any rebuttal to their solution, and judges award points. There is more to it, but in my experience students *love* the experience. Mounting a team requires training and practice, as does any debate program.

The benefits? Not only do students get to solve hard problems, but they do it in a competitive team setting. In a wrangle, no one student may present a solution more than once—everyone on the team must understand the solutions. Students must listen carefully to the opposing team's presented solutions, since someone on their team will have to critique that solution for correctness and thoroughness of presentation. Each student must learn to present a mathematical solution clearly and using proper mathematical language. To not do so risks losing points to the opposing team or to the judges.

Competitions come in many forms
Some are for individuals and others for teams. Many have tight time limits, and I am not fond of these, however some kids thrive on this sort of thing. And despite some initial reluctance, many students find that they enjoy such.

Some competitions are filled with exercises not very different from those found in textbooks, but they are stripped of the context (which technique(s) should I use?) a text provides. Others are true problems that are not something you can instantly solve if you know some method. Still others are long, multi-step puzzles that take deep and lengthy thought (the amazing Mandelbrot team problems are an example).

What do competition problems look like?
There is no simple answer to this! The competitions vary so much in format that all one can do is give examples.

Getting started
Most of us did not participate in math competitions as kids, so we naturally feel no small trepidation jumping into leading students. Help is available! Please reach out to me or to Geoffrey Moon for help getting started.

Addressing student reluctance, selling participation
It is usually not enough to tell kids that nothing is at stake, no grades are involved—though you could start there. I think getting them hooked on puzzles is a better way. We'll discuss this further in our workshop session, with insights on running MOEMS in Santa Fe schools from the District's Geoffrey Moon.

Most important? Practice and preparation
Students performing well in math classes often struggle with true problem competitions. They must practice new ways of thinking. I'll be talking about this in detail during the workshop session. Mostly it comes down to introducing problem-solving techniques through doing lots of problems as a group.

15 Problem Solving Strategies[1]

[1] Inspired by James Tanton, https://www.maa.org/math-competitions/teachers/curriculum-inspirations

1. Engage in Successful Flailing
2. Do Something
3. Engage in Wishful Thinking
4. Draw a Picture
5. Make it small! (smaller version of prob.)
6. Enumerate cases or make a table
7. Work backwards
8. Eliminate Incorrect Choices
9. Perseverance
10. Second-Guess the Author
11. Avoid Hard Work
12. Go to Extremes
13. Name and conquer (let n=…)
14. Use manipulatives or relevant physical objects
15. Test cases with your own numbers

<u>Some sample problems from various competitions</u>

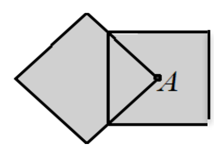
As you look through these problem sets, please reflect not only on the solution, but on whether your students can solve them—why or why not—and what tools and practice they would need to succeed with them.

Starter problem: Igor and his two friends played chess. Everyone played two games. How many games were played?

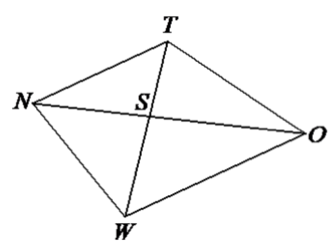
NOTE: No use of calculators or computers! None of these problems need it!

MOEMS Elementary and Middle Problems, Fall 2021:

1. What whole number is equal to $16 - \sqrt{32 - \sqrt{49}}$?



2. In the partial sequence ..., 987, N, 2584, 4181, ..., each new term is the sum of the two previous terms. Find the whole number value of N.



3. Two congruent squares overlap, as shown, so that vertex A of one square lies at the intersection of the diagonals of the other square. The side of each square has length 12 inches. Find the number of square inches enclosed by the shaded region.

4. A network of roads connects locations *T*, *O*, *W*, *N*, and *S* as shown. How many different routes are there to travel from T to W along the given roads (line segments) without retracing any part of the route or going to any town more than once per route?

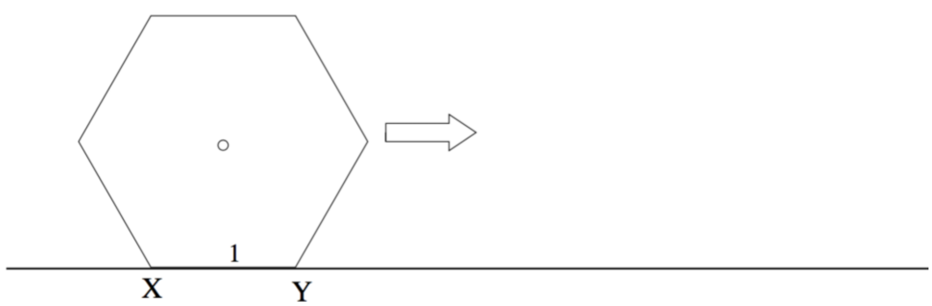
5. How many quadrilaterals can be outlined by selecting two or three of the four smaller congruent equilateral triangles shown at the right without moving any of the triangles? Congruent but non-identical quadrilaterals are considered different.

UNM PNM Math Contest (grades 7-12, details later in this document)

1. What is the last digit of 2003^{2003}?

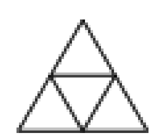
2. Recall that n! is the product of the first n positive integers, that is 2!=2•1, 3!=3•2•1, etc.
   a. How many zeroes are at the end of 17!?
   b. What is the smallest such n such that n! ends in exactly 37 zeroes?

3. A regular hexagonal wheel is resting on side XY, as in the diagram below. All sides have a length of 1 unit. Suppose the hexagon is rolled forward (without slipping) along a straight line until the side XY is again back on the line (for the first time since it started to roll).
   a. What is the length of the path traveled by the center (or axle)?
   b. What is the length of the path traveled by vertex Y?



4. I invite 10 couples to a party at my house. I ask everyone present, including my wife, how many people they shook hands with. It turns out that everyone shook hands with a different number of people. If we assume that no one shook hands with his or her partner, how many people did my wife shake hands with? (I did not ask myself any questions.)

5. A spider is standing at the center of the bottom of a glass. The spider wants to reach a delicious ant that is standing on the rim of the glass. Assume the spider walks at constant speed and the ant, unaware of the danger, does not move.



   a. Suppose the glass is cylindrical of radius 1 unit and height 2 units. What distance should the spider walk to have her meal as quickly as possible?
   b. Suppose the glass now has a square base of side 2 units and height 2 units. The ant is standing in one of the top corners of the glass, and the

spider is still at the center of the base. What distance should the spider walk to have her meal as quickly as possible?

Discussion, Q & A, Things to Think About

- Especially, how would you get started with competition problems?
- Which of these could you use at the grade level you teach?
- Where would you place the grade level of each problem?
- Discuss your personal stories around problems that look like this, if any?
- Reflect on issues of language fluency (native English and ELA) and these problems

A partial list of competitions

- Math Olympiads for Elementary & Middle Schools (MOEMS) https://moems.org
- UNM-PNM Math Contest http://mathcontest.unm.edu
- Mandelbrot Competition (discontinued) http://mandelbrot.org/
- American Mathematics Contest (AMC) 8, 10 & 12 https://www.maa.org/math-competitions
- MathCounts https://www.mathcounts.org

SFPS Math Matters Framework:
https://docs.google.com/presentation/d/1GMtr40vErG4R0sBr6UgrEouXSNqdqs8yEbPhIsQh8Vc/edit#slide=id.g84160ce794_0_59
SFPS Priority Math Standards:
https://docs.google.com/document/d/1PcT9fWUkXFQcZNIpSWLAneSLmd8vjtBaRhEa9Fj3iE8/edit
SFPS Elementary school math curriculum:
SFPS EnVision math curriculum from Erica Wheeler at
savvasrealize.com ; Username: SFPSVolunteers Password: Welcome1

Common Core Connections

Standards for Mathematical Practice: All of them!

- MP1: “Make sense of problems and persevere in solving them”.
- MP2: “to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own”
- MP3: “Construct viable arguments & critique the reasoning of others”
- MP4: “Model with mathematics”
- MP5: “Use appropriate tools strategically”
- MP6: “Attend to precision”
- MP7: “Mathematically proficient students look closely to discern a pattern or structure”
- MP8: “Look for and express regularity in repeated reasoning”

Standards for Mathematical Content

Since each problem uses a mix of nearly all of the grade level standards—and nearly all of them show up in one form or another (or could, depending on the strategies used) throughout these problems—no specific list will appear here.

Paul Zeitz on exercises and problems[2]
An exercise is a question that tests the student's mastery of a narrowly focused technique, usually one that was recently "covered." Exercises may be hard or easy, but they are never puzzling, for it is always immediately clear how to proceed. Getting the solution may involve hairy technical work, but the path towards solution is always apparent. In contrast, a problem is a question that cannot be answered immediately. Problems are often open-ended, paradoxical, and sometimes unsolvable, and require investigation before one can come close to a solution. Problems and problem solving are at the heart of mathematics. Research mathematicians do nothing but open-ended problem solving. In industry, being able to solve a poorly-defined problem is much more important to an employer than being able to, say, invert a matrix. A computer can do the latter, but not the former.

A good problem solver is not just more employable. Someone who learns how to solve mathematical problems enters the mainstream culture of mathematics; he or she develops great confidence and can inspire others. Best of all, problem solvers have fun; the adept problem solver knows how to play with mathematics, and understands and appreciates beautiful mathematics.

An analogy: The average (non-problem solver) math student is like someone who goes to a gym three times a week to do lots of repetitions with low weights on various exercise machines. In contrast, the problem solver goes on a long, hard backpacking trip. Both people get stronger. The problem solver gets hot, cold, wet, tired, and hungry. The problem solver gets lost, and has to find his or her way. The problem solver gets blisters. The problem solver climbs to the top of mountains, sees hitherto undreamed of vistas. The problem solver arrives at places of amazing beauty, and experiences ecstasy which is amplified by the effort expended to get there. When the problem solver returns home, he or she is energized by the adventure, and cannot stop telling others about their wonderful experiences. Meanwhile, the gym rat has gotten steadily stronger, but has not had much fun, and has little to share with others.

[2] *The Art and Craft of Problem Solving*, 1999

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Introduction to the UNM-PNM Math Contest (mathcontest.unm.edu)

In 1966 the first UNM-PNM Math contest was held. Although this will be the 51st year of the competition, many math teachers in New Mexico have never heard of it! From the contest website:

> *The goal of the contest is to promote mathematics education in New Mexico by rewarding students, teachers, and their schools for mathematics excellence. Between 700 to 1200 New Mexico students benefit from this program annually. The contest is open to all students in grades 7 - 12 as well as interested students in lower grades. The contest has two rounds of exams designed to test mathematical potential and ingenuity as well as formal knowledge. Round I is administered at the students' home school. The top finalists are invited to the UNM Campus in early February to compete in Round II. The winners will be determined using the number of participants with non-zero scores on the Contest First Round scaled proportionately to the school size.*

Teachers register at the program's website, and can access many years of the Contest's tests, along with fully worked (sometimes using multiple approaches) solutions to the problems.

This contest is one of my favorites, and one that is well suited to New Mexico. It can lead to statewide recognition for our students. In my opinion, it is best not to think of the test as a competition, but rather as a way to expose students to great problems, and problems with which you are only permitted to use two tools—paper and pencil.

The first round is conveniently held at your own school (though some schools in the same district partner and proctor it in a single location). There are typically ten problems, and students have up to three hours to complete the test. Students seldom have any experience with such a long test. They are openly skeptical of even trying this approach out. I often received student comments, post-test, that it was great to have so much time to think about the problems—that three hours was perfect. How often do students have three uninterrupted hours to focus on anything, particularly mathematics? There is none of the "rush to complete" pressure that is common to so many math tests and competitions.

What are the problems like? They vary in levels and difficulties. All students take the same test. Students are ranked at grade level. A couple of problems will contain trigonometry or logarithms—things the younger kids will not have seen—but most are accessible to middle school math (though some involve basic terms and notation unfamiliar to most of the students and these are easy to introduce in practice sessions) with some further preparation and practice. Younger students will likely not complete many of the problems—and that's okay. Most of the problems depend more on problem solving chops than on mastery of techniques. Many of your "best" math students (best grades) will be utterly stumped by the problems. Students who have learned how to think deeply about math, persevere, and know what sorts of strategies will help them dig into problems will do much better.

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A Math Wrangle Team Guide

In a Math Wrangle, the judges are looking for solutions, NOT answers.
A solution includes an answer, but also includes an explanation of how you found the solution, patterns you noticed along the way, and why you think that your answer is correct.

When rebutting the other team (critiquing the other team's solution), you might explain something they forgot to explain, or show how you approached the problem differently. You might also explain why they are wrong, show another solution, or prove that you have a better solution.

Things to Discuss While Preparing for the Wrangle

- Are you certain your answers are right?
- Can you explain why and prove your answers?
- Prepare things you might say to critique the other team.
- Does your team have a way to go above and beyond with any problems? This might be to:
    - Solve something not asked
    - Find a general rule or pattern
    - Demonstrate a way to solve the problem more easily
    - Extend the problem

Some suggestions

- Remember that this is a team effort. Read through each problem carefully and solve them in team discussion.
- Keep track of time. We will remind you halfway through and when 5 minutes remain.
- Leave enough time to discuss your team's presentations of the problems.
- Think about what presenting your solutions at the wrangle will look like, how you will use the board, etc. All of you should have some understanding of all of the problems.
- Think about which problems your team would like to challenge the other team to solve.

Standard Math Wrangle Rules:
http://sigmaa.maa.org/mcst/documents/math_wrangle_rules_revised_9feb12.pdf

- Note that wrangles may take a variety of forms and James has experience with several of these. Please reach out to him with questions.